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Introduction to "Towards the Ultimate Conservative Difference Scheme. V. A Second-Order Sequel to Godunov's Method"

BRAM VAN LEER'S QUEST FOR PERFECTION

The paper reproduced in this special issue is the fifth and last of a series, started in 1973 (Refs. [1] to [5]), with the general objective of a march "Towards the Ultimate Conservative Difference Scheme." This is a unique example of a persistent pursuit, spanning five or six years, of a clearly stated objective, although the outcome was not initially known to the author, which led to the foundation of modern CFD methodology. In commenting on this fifth "installment" it is difficult to dissociate it from the four previous ones, as they form a consistent set, although the first four papers of the series deal essentially with linear convection. When B. Van Leer started this work, in the early seventies, the development of numerical schemes for the compressible flow equations had already reached an advanced stage with the availability of the second-order centered scheme of Lax and Wendroff (LW) [6] and its two-step variant introduced by MacCormack [7, 8]. The latter was a major step forward, since it simplified considerably the formulation (avoiding computations of the Jacobians and requiring only flux evaluations) and opened the way to practical applications of the LW scheme, leading to the first significant computations of two- and three-dimensional, shock-capturing, inviscid and viscous flows on complex geometries. Actually, the first practical application of the finite volume method [9] was based on MacCormack's scheme.

It was clearly recognised at that time that the second-order centered LW scheme generated oscillations around shocks. These were accepted as a "nuisance" which had to be filtered out by the addition of higher order dissipation terms, or artificial viscosity terms, a concept already introduced by Von Neumann [10] and also analysed by Lax and Wendroff. Based on empirical and intuitive arguments, MacCormack and Paullay [11] introduced a more sophisticated form for the dissipation terms involving a second difference of the pressure as a detector of high gradients, multiplying second-order differences of the basic variables. This appeared to be very effective essentially, as is known today from B. Van Leer's work, because of the nonlinearity introduced hereby.

Working as an astronomer, with the task of simulating the formation of stars and stellar systems, a task somewhat isolated from the main pressure of aerospace applications, B. Van Leer initiated a fundamental analysis of the main properties needing to be satisfied by the "ultimate" numerical schemes, namely monotonicity and conservation coupled to second- or higher order accuracy. Within this context, the development of Bram's work over the six-year duration of his "ultimate" series is remarkable. In a free and creative spirit he undertook a return to the fundamentals, selecting the notion of monotonicity as the basis of the foundation of numerical schemes. Initiating the analysis with the LW and Fromm schemes applied to the linear one-dimensional convection equation [1, 2], he introduced in the two initial papers the fundamental concept of slope limiters. Although a similar concept had been introduced, at nearly the same time, by Boris and Book [12], Van Leer's approach is distinctive in separating the update procedure into an interpolation or reconstruction step followed by an evolution step. This separation greatly clarifies the correct treatment of systems of equations on the basis of a scalar analysis. This major contribution established clearly for the first time that the way around the limitations expressed by Godunov's Theorem [14], linking monotonicity to first-order accuracy for linear schemes, was to introduce nonlinear contributions in the scheme in the form of limiters. As these limiters require upstream information, it became clear that the way to the "ultimate" scheme was to look for upwind-biased schemes of at least second-order accuracy. Since the well-known first-order upwind CIR scheme [13] cannot be made conservative in a straightforward way, Van Leer turned to the largely ignored work of Godunov [14]. This is another historical merit of the fourth and fifth papers of the series, namely the recognition and extension of Godunov's fundamental new approach to numerical methods for hyperbolic conservation laws, characterized by the introduction of physical, simple, solutions of the flow equations to the numerical scheme. In the present context, these exact solutions of the inviscid conservation laws describe the time evolution of an initial, one-dimensional discontinuity, known as the Riemann problem. The fourth paper of the series, although applied to the one-dimensional linear convection equation, sets the basis of modern upwind-biased schemes of second- (or third-) order accuracy, monotone and conservative, based on piecewise linear, but limited, variations of the solutions, coupled to the exact solution of the cell interface discontinuities, following Godunov's approach. It also con-

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firms the excellent properties of a continuous limiter, based on a harmonic average of gradients, known today in the literature as the Van Leer limiter.

These developments then culminate in the fifth paper of the series which focuses exclusively on the one- and two-dimensional equations for compressible gas dynamics. Although developed for a Lagrangian formulation, which is not much favoured nowadays, this paper has had profound influence on all of the subsequent work related to upwind schemes. The acronym, MUSCL, of the code developed by P. Woodward along the lines set by B. Van Leer is largely used today to identify second-order upwind methods based on linear extrapolation of variables, applied with exact or approximate Riemann solvers. Although exact solutions of the Riemann problem are used in the paper, the opening towards approximate solutions is already implied. Two-dimensional applications are treated in this work through a Strang-type time splitting, relying on a superposition of one-dimensional Riemann solutions. We now draw the reader's attention to the conclusions of this paper and to the outlook for the future it contains, in light of today's state of the art. As in many landmark papers, it is stimulating to read and learn from the vision and lucidity with regard to some fundamental issues as they appeared in their initial development. Today, there is no doubt that the "upwind" concepts, initiated by Godunov and further extended and developed by Van Leer, are indeed the ultimate components of any high order scheme.

A few weeks ago (May 1–2, 1997), a two-day "Godunov Symposium," which was organized by B. Van Leer and which will be reported in this journal, was held at the Michigan State University in Ann Arbor in honor of S. K. Godunov, who was present and commented on the historical development of his work. It was indeed extremely gratifying that 45 years after Godunov's initial work in the former Soviet Union and 20 years after Van Leer's second-order extension was developed at the University of Leiden in the Netherlands, the circle was closed through this historical symposium held in the U.S.A. Two of the questions raised in this last paper of Van Leer's series are still very actual and are still not fully resolved: the application of genuinely multidimensional upwinding and the extension to problems with more complex physical properties, such as magnetohydrodynamics (MHD). These were high on the list of topics discussed at this symposium and actually lead to a more general question with regard to the fundamental approach to simulating complex flow systems through the introduction of simple solutions in the algorithm: How much of the known physics is it necessary to introduce in order to realize a substantial gain in accuracy? The success of approximate Riemann solvers shows that the question is meaningful, while the improvements over dimensional split methods obtained with multidimensional upwinding are significant for first-order schemes, but are not yet fully established for general 3D flows at the level of second-order accuracy. Similar questions can be raised for the MHD applications, showing how actual the work presented and the issues raised in B. Van Leer's paper still are.

The quest for perfection can never end.

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